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Heun functions and the energy spectrum of a charged particle on a sphere under a magnetic field and Coulomb force

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Abstract
We study the competitive action of magnetic field, Coulomb repulsion and space curvature on the motion of a charged particle. The three types of interaction are characterized by three basic lengths: the magnetic length l_B, the Bohr radius l_0, and the radius of the sphere R. The energy spectrum of the particle is found by solving a Schrödinger equation of the Heun type, using the technique of continued fractions. This displays a rich set of functioning regimes where ratios $\frac{R}{l_B}$ and $\frac{R}{l_0}$ take definite values.

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1. Introduction

The motion of charged particles under a constant magnetic field on a sphere has been introduced by Haldane [1] and several other authors [2–4] to discuss the fractional quantum Hall effect, and to compute several relevant quantities for a system of N independent particles. The sphere is basically a macroscopic support for the motion of a large assembly of particles and its radius R is usually of macroscopic order.

However, in recent years, it has been possible to realize spheres of nanostructures ranging from fullerene size to spherical nanosized objects such as SiO_2 balls in opals [5, 6]. The motion of charged particles on such spheres with a constant magnetic field parallel to a given diameter of the sphere has been studied quite extensively [7–9]. The energy spectrum helps us to understand some interesting aspects, such as orbital magnetism.

In view of future developments, it appears useful to consider the motion of charged particles on a sphere under the action of competing forces. Besides the curvature of the configuration space and the confining effect of the radial magnetic field, it may be interesting to introduce the Coulomb electric force due to an ion positioned somewhere on the sphere. This situation may arise in the process of fabricating the actual sphere when a charged impurity may slip into the spherical surface and become a centre of interaction [10]. The point is thus to see how such a charged impurity affects the energy spectrum and whether or not such effects are, in practice, relevant. Moreover, the choice of a magnetic monopole generating a radial magnetic field is made to obtain analytical solutions. Since a constant magnetic field parallel to a given diameter does not allow us to find an exact wavefunction, we replace it with a radial field in which competing effects are well described. The strength and range of these effects can be described by three characteristic lengths:

- the magnetic confinement length, $l_B = \sqrt{\frac{\hbar}{eB}}$;
- the Bohr radius of the hydrogen atom, $l_0 = \sqrt{\frac{\hbar^2}{M\kappa e^2}}$;
- the radius of the sphere, R.

Here, M and e are respectively the mass and the charge of the particle, B is the constant radial magnetic field on the sphere, and $\kappa = \frac{1}{4\pi\epsilon_0}$ in MKSA units.

Problems with the three characteristic lengths do occur in physics. We may cite the case of one-dimensional superconductivity in the presence of twinning defects where electron pairs move in a constant magnetic field and a finite Dirac Comb [11] potential. The three characteristic lengths involved are

- the coherence length of the electron pairs, l_c;
- the magnetic confining length, $l_B = \sqrt{\frac{\hbar}{eB}}$;
- the period of the Dirac Comb potential, d.

In the present case, we look for stationary states of a charged particle on a sphere under a constant radial magnetic field, which are repelled by a point charge placed on the sphere. This is done by solving the time-independent Schrödinger equation. The solution turns out to be expressible in terms of a generalization of a Gauss hypergeometric function called the Heun function [12]. This new function has been studied by several authors [12–14]. Only one type of Heun function turns out to be relevant for our problem. In fact, to fulfil the requirements of quantum mechanics, it is necessary to demand that the Heun function be defined all over the sphere, and also that it be of square integrable class. This yields a condition on one of the coefficients of the Heun equation, which leads to the quantization of the energy levels.

In section 2 we present the classical aspects of the dynamics showing that, in particular, with the motion in the θ-direction the energy should be larger than a certain limit. In section 3 we introduce the quantum treatment which leads to the Heun equation. We compute in section 4 the $R \to \infty$ and $S \to \infty$ limit, which describes the recovery of the motion in planar geometry. In this case, the limiting problem is that of the relative motion of two equally charged particles on plane in a uniform perpendicular magnetic field and under Coulomb repulsion. Our results and comments are contained in section 5.

2. Classical considerations

Let (θ, ϕ) be the angular coordinates of a particle of mass M and charge e on a sphere of radius R. There exists a constant radial magnetic field $\vec{B}$ pointing outwards on the surface of the sphere. $\vec{B}$ is described by the vector potential [1]

$$\vec{A} = -\Phi_0 \frac{S}{2\pi R} \cot\theta \vec{\phi} \tag{1}$$

where S (the half magnetic flux) is the strength of the magnetic monopole generating the magnetic field $\vec{B}$, $\Phi_0 = \frac{h}{e}$, the elementary flux unit, and $\vec{\phi}$ is the unit vector along the ϕ-direction on the sphere. In fact, we have

$$\vec{\nabla} \times \vec{A} = \vec{B} = (B_r = B, B_\theta = B_\phi = 0). \tag{2}$$

We assume, in addition, the existence of a repulsive Coulomb interaction due to a charge e, placed at the north pole of the sphere, which is given by the potential

$$V(\theta) = \kappa \frac{e^2}{2R \sin \frac{\theta}{2}}. \tag{3}$$

The dynamics of the particle is given by the Hamiltonian

$$H = \frac{p_\theta^2}{2MR^2} + \frac{(p_\phi + \hbar S \cos\theta)^2}{2MR^2 \sin^2\theta} + \kappa \frac{e^2}{2R \sin \frac{\theta}{2}} \tag{4}$$

where p_θ and p_ϕ are the canonical conjugate momenta to θ and ϕ. The motion at constant energy is defined by

$$H = E. \tag{5}$$

In order to simplify the discussion, we work with dimensionless quantities, Π_θ, Π_ϕ and ϵ, defined by

$$p_\theta = \hbar \Pi_\theta \qquad p_\phi = \hbar \Pi_\phi \qquad E = \frac{\hbar^2}{2MR^2}\epsilon. \tag{6}$$

Introducing the Bohr radius

$$l_0 = \frac{\hbar^2}{M\kappa e^2} \tag{7}$$

the equation of energy conservation becomes

$$\Pi_\theta^2 + \frac{(\Pi_\phi + S\cos\theta)^2}{\sin^2\theta} + \frac{R}{l_0}\frac{1}{\sin\frac{\theta}{2}} = \epsilon. \tag{8}$$

We see the competing roles of geometry and Coulomb repulsion expressed by the ratio $\frac{R}{l_0}$.

Moreover, in spherical coordinates, we have

$$\begin{aligned} B_r &= (\vec{\nabla} \times \vec{A})_r = \frac{-1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\Phi_0 S}{2\pi R}\cot\theta\right) \\ B_r &= \Phi_0 \frac{S}{2\pi R^2} = |\vec{B}| = B. \end{aligned} \tag{9}$$

We can then introduce a magnetic length $l_B = \sqrt{\frac{\hbar}{eB}}$ and express S as

$$S = \frac{R^2}{l_B^2}. \tag{10}$$

Hence, the combined interactions can be expressed through the ratios of lengths $\frac{R}{l_B} = \sqrt{S}$ and $\frac{R}{l_0}$. Note that, since $0 < \theta < \pi$, $\sin\frac{\theta}{2}$ is always positive and the left-hand side of equation (8) is the sum of three positive terms.

Now, since p_ϕ is a conserved quantity (ϕ is a cyclic variable) one may set $p_\phi = m\hbar$, with m fixed equal to Π_ϕ. The orbit in the phase space (Π_θ, θ) is a curve of equation

$$\Pi_\theta^2 + \frac{1}{\sin^2\theta}\left(m + \frac{R^2}{l_B^2}\cos\theta\right)^2 + \frac{R}{l_0}\frac{1}{\sin\frac{\theta}{2}} = \epsilon. \tag{11}$$

Equation (11) can be viewed as the motion of a fictitious one-dimensional particle in θ-space. The potential function for $0 < \theta < \pi$ has a minimum value ϵ_0 which depends on m, $\frac{R}{l_B}$ and $\frac{R}{l_0}$, thus the motion is only possible if $\epsilon > \epsilon_0$.

The action for a complete cycle is J_θ given by

$$J_\theta = 2\hbar \int_0^{2\pi} \mathrm{d}\theta \sqrt{\epsilon - \frac{1}{\sin^2\theta}\left(m + \frac{R^2}{l_B^2}\cos\theta\right)^2 - \frac{R}{l_0}\frac{1}{\sin\frac{\theta}{2}}}. \tag{12}$$

The Bohr–Sommerfeld quantization of the θ-motion by setting

$$J_\theta = h\left(n + \tfrac{1}{2}\right) \qquad n \in \mathbb{N} \tag{13}$$

yields the quantized energy levels through $\epsilon = \epsilon_{n,m}$ such that

$$\int_0^{2\pi} \sqrt{\epsilon - \frac{1}{\sin^2\theta}\left(m + \frac{R^2}{l_B^2}\cos\theta\right)^2 - \frac{R}{l_0}\frac{1}{\sin\frac{\theta}{2}}}\,\mathrm{d}\theta = \pi\left(n + \frac{1}{2}\right). \tag{14}$$

Unfortunately the integral cannot be evaluated in closed form in terms of simple functions. This aspect will not be treated here.

3. Quantum motion

The Hamiltonian of the motion can be suggestively re-expressed in terms of the operators $\vec{\Lambda} = \vec{r} \times (\vec{p} + e\vec{A})$ as in [1]

$$H = \frac{\vec{\Lambda}^2}{2MR^2} + \kappa\frac{e^2}{2R\sin\frac{\theta}{2}}. \tag{15}$$

$\vec{\Lambda}$ has Cartesian components

$$\begin{pmatrix}\Lambda_x\\ \Lambda_y\\ \Lambda_z\end{pmatrix} = \begin{pmatrix} M_x + \hbar S\cos\phi\frac{\cos^2\theta}{\sin\theta}\\ M_y + \hbar S\sin\phi\frac{\cos^2\theta}{\sin\theta}\\ M_z - \hbar S\cos\theta\end{pmatrix} \tag{16}$$

with usual definition of $\vec{M}$

$$\begin{pmatrix}M_x\\ M_y\\ M_z\end{pmatrix} = \mathrm{i}\hbar\begin{pmatrix}\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\\ -\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\\ -\frac{\partial}{\partial\phi}\end{pmatrix}. \tag{17}$$

Consequently, with

$$M^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{18}$$

one has

$$\Lambda^2 = M^2 + \hbar^2\frac{S^2 + 2\mathrm{i}S\cos\theta\frac{\partial}{\partial\phi}}{\sin^2\theta} - \hbar^2 S^2 \tag{19}$$

and the Schrödinger equation for stationary states $\Psi(\theta,\phi)$ reads

$$\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right)\Psi(\theta,\phi) + \frac{\left(\frac{\partial^2}{\partial\phi^2} - S^2 - 2\mathrm{i}S\cos\theta\frac{\partial}{\partial\phi}\right)}{\sin^2\theta}\Psi(\theta,\phi) + \left[S^2 - \frac{R}{l_0}\frac{1}{\sin\frac{\theta}{2}} + \epsilon\right]\Psi(\theta,\phi) = 0. \tag{20}$$

As ϕ is a cyclic variable, p_ϕ is conserved, thus the wavefunction $\Psi(\theta,\phi)$ can be set in the form

$$\Psi(\theta,\phi) = \mathrm{e}^{\mathrm{i}m\phi} F(\theta), \tag{21}$$

with $m \in \mathbb{Z}$. We get a new equation for $F(\theta)$

$$F''(\theta) + \cot\theta F'(\theta) + \left\{ -\frac{m^2 + S^2 - 2mS\cos\theta}{\sin^2\theta} - \frac{R}{l_0}\frac{1}{\sin\frac{\theta}{2}} + \epsilon + S^2 \right\} F(\theta) = 0. \tag{22}$$

The choice of m as a relative integer guarantees the uniformity of the wavefunction under ϕ-rotation. We seek the solution $F(\theta)$ which vanishes at $\theta \to 0$ because of the Coulomb repulsion at the north pole and which is square integrable.

We now transform this equation with the change of variable $z = \cos\theta$ to bring it back to a known canonical form. F is now a function of z, satisfying

$$(1 - z^2)F''(z) - 2zF'(z) + \left\{ -\frac{m^2 + S^2 - 2mSz}{1 - z^2} - \frac{\sqrt{2}R}{l_0}\frac{1}{\sqrt{1 - z}} + (\epsilon + S^2) \right\} F(z) = 0. \tag{23}$$

A change of unknown function through the substitution

$$F(z) = (1 - z)^{\frac{a}{2}}(1 + z)^{\frac{b}{2}} P(z) \tag{24}$$

where a and b are free parameters to be chosen later, and a last change of variable $x = \sqrt{\frac{1-z}{2}}$ transform equation (23) into a Heun equation in its canonical form

$$\begin{aligned}(x - 1)(x - a')P''(x) &+ \left[(\epsilon' + \delta + \gamma)x - (\epsilon' + a'\delta + \gamma(1 + a')) + \frac{a'\gamma}{x}\right] P'(x) \\ &+ \left[\alpha\beta - \frac{\alpha\beta h}{x}\right] P(x) = 0\end{aligned} \tag{25}$$

with the relation $\alpha + \beta + 1 = \gamma + \delta + \epsilon'$.

This equation is a second-order equation with four regular singularities, located at ($x = 0, 1, a'$ and ∞). h is called the 'auxiliary parameter' and is significant for our problem. The solution to this Heun equation is a local Heun function admitting an expansion around $x = 0$. Such a solution has a convergent expansion in a domain centred at $x = 0$ and for $|x| < 1$ and $|x| < |a'|$. The south pole which corresponds to $x = 1$ will be a singular point since the expansion is divergent there.

To obtain equation (25), the coefficients a and b must fulfil the following conditions:

$$a^2 - b^2 = -4Sm \qquad a^2 + b^2 = S^2 + m^2. \tag{26}$$

As $F(\theta)$ must not have singularity as $\theta \to 0$ (or $\theta \to \pi$) and equivalently $P(x)$ must be regular for $x \to 0$ (or $x \to 1$), one must choose a and b positive, i.e. with $S = \frac{R^2}{l_B^2}$:

$$a = |S - m| \qquad b = |S + m|. \tag{27}$$

The parameters of the Heun equation are then related to the parameters of the problem through

$$\begin{aligned} a' &= -1 \\ \gamma &= 2a + 1 \\ \delta &= \epsilon' = b + 1 \\ \alpha\beta &= (a + b)(a + b + 2) - 4(\epsilon + S^2) \\ \alpha\beta h &= -4\frac{R}{l_0}. \end{aligned} \tag{28}$$

One possibility to make our solution valid everywhere on the sphere is to reduce the Heun function down to a polynomial. But this procedure imposes a condition which makes the radius of the sphere discrete. This is, in general, physically not acceptable.

The other possibility is to turn to the procedure of augmented convergence [12] which makes use of an expansion in terms of hypergeometric functions [13, 14]. Following Ronveaux, we first expand our local Heun function $P(x)$ in terms of hypergeometric functions [12]

$$P(x) = \sum_\nu c_\nu y_\nu(x) = \sum_\nu c_\nu F(-\nu, \nu + \delta + \gamma - 1; \gamma; x) \tag{29}$$

where $F(-\nu, \nu + \delta + \gamma - 1; \gamma; x)$ is a local solution of a hypergeometric equation which matches Heun's solution at the singularities 0 and 1 (the same exponents at the singularities 0 and 1). As $P(x)$ obeys the Heun equation, c_ν fulfils the three-way recursion relation [12]

$$K_\nu c_{\nu-1} + L_\nu c_\nu + M_\nu c_{\nu+1} = 0 \tag{30}$$

where

$$\begin{aligned}
K_\nu &= \frac{(\nu+\alpha-1)(\nu+\beta-1)(\nu+\gamma-1)(\nu+w-1)}{(2\nu+w-1)(2\nu+w-2)} \\
L_\nu &= \alpha\beta h + a'\nu(\nu+w) - \frac{\epsilon'\nu(\nu+w)(\gamma-\delta) + [\nu(\nu+w)+\alpha\beta]\,[2\nu(\nu+w)+\gamma(w-1)]}{(2\nu+w-1)(2\nu+w+1)} \\
M_\nu &= \frac{(\nu+1)(\nu+w-\alpha+1)(\nu+w-\beta+1)(\nu+\delta)}{(2\nu+w+1)(2\nu+w+2)}
\end{aligned} \tag{31}$$

with $w = \gamma + \delta - 1$.

The behaviour of $\{y_\nu\}$ is given by [13]:

$$\lim_{\nu\to\infty} \left|\frac{y_{\nu+1}}{y_\nu}\right| = \left|\frac{1+X}{1-X}\right| \tag{32}$$

where $X = \sqrt{1 - x^{-1}}$.

To determine this limit, we must find the asymptotic behaviour of $\{c_\nu\}$. For $\nu \to \infty$, the coefficients of the recursion relation become

$$K_\nu \to \tfrac{1}{4}\nu^2 \qquad L_\nu \to -\left(\tfrac{1}{2} - a'\right)\nu^2 \qquad M_\nu \to \tfrac{1}{4}\nu^2. \tag{33}$$

Equation (30) tends to the critical equation $\rho^2 + 2(2a' - 1)\rho + 1 = 0$ (where $c_{\nu+i}$ has been replaced by ρ^i) which admits the roots ρ_1 and ρ_2 given by

$$|\rho_1| = \left|\frac{1-A}{1+A}\right| \qquad |\rho_2| = \frac{1}{|\rho_1|} \tag{34}$$

where $A = \sqrt{1 - a'^{-1}}$. Erdélyi [13] has shown that the convergence of the series occurs in a region determined by

$$\left|\frac{1+X}{1-X}\right| < \frac{1}{|\rho_1|} = |\rho_2|. \tag{35}$$

This is the so-called augmented convergence phenomenon. This inequality implies that the point x lies in the interior of the ellipse $\mathcal{E}$ with foci at 0 and 1. Thus, our wavefunction $P(x)$ is now well defined on each pole of the sphere.

The condition for which the solution of the Heun equation has augmented convergence at $x = 0$ and $x = 1$ (poles of the sphere) is that the parameter h must be a zero of the continuous fraction [15] obtained from the three-way recursion relation [13], i.e.

$$B_0 + \cfrac{A_1}{B_1 + \cfrac{A_2}{B_2 + \cfrac{A_3}{B_3 + \cdots}}} = 0 \tag{36}$$

where

$$A_\nu = -\frac{K_\nu}{M_\nu} \qquad \text{and} \qquad B_\nu = \frac{L_\nu}{M_\nu}. \tag{37}$$

Hence, for $h = h_n$, we can construct a bona-fide wavefunction for a stationary state. The set of h_n can be obtained by numerical treatment of the continuous fraction. We use the so-called modified Lentz's method [16], calculating by iterations the values of the continuous fraction and stopping when a given precision is obtained.

The energy spectrum of the problem is of the form

$$\epsilon = \epsilon_n = \frac{R}{l_0}\frac{1}{h_n} + \left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}+1\right) - \frac{R^4}{l_B^4} \tag{38}$$

with $n = 1, 2, \ldots, \infty$. Now, for $S = \frac{R^2}{l_B^2}$ and m given, all the parameters of the Heun equation are defined and we can find, using classical numerical methods, the solution in the form of the transcendental equation (36). Note that, since we are interested in the energy spectrum, we have replaced h_n in equation (36) by the following expression

$$h_n = \frac{4R}{l_0\left[4\left(\epsilon_n + \frac{R^4}{l_B^4}\right) - (a+b)(a+b+2)\right]}. \tag{39}$$

This allows us to compute directly the energy values ϵ_n, which is done in section 5.

4. The $R, S \to \infty$ limit

It is interesting to look at the $R, S \to \infty$ limit, since it is relevant for many problems in physics. In the fractional quantum Hall effect, for simplicity many calculations were made in spherical geometry [1], then the physical effects studied in planar geometry. For example, the computation of the energy of the neutralizing background takes a very simple form in spherical geometry, since it amounts to putting a particle of opposite charge at the centre of the sphere. In this limit, the curvature of the sphere vanishes and we expect to recover the plane. We know that S and R are related by equation (10), thus when $R \to \infty$, S must go also to infinity. For physical reasons, S must be infinite, because there is a constant perpendicular magnetic field (i.e. a non-vanishing flux) on the infinite plane.

Starting with the Schrödinger equation on the sphere, after the substitution (24), one has

$$(1-z^2)P''(z) + [b - a - (a+b+2)z]\,P'(z) + \left[\epsilon + S^2 - \left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}+1\right) - \frac{\sqrt{2}R}{l_0\sqrt{1-z}}\right]P(z) = 0. \tag{40}$$

Thus we look for a change of variable to describe the plane. Such a variable can be $\chi = 2S(1-z)$ and transforms equation (40) into

$$4S\left\{\chi P''(\chi) + \left(a + 1 - \frac{a+b}{4S}\chi\right)P'(\chi)\right\} + 4S\left\{\left[\frac{\epsilon + S^2 - \left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}+1\right)}{4S} - \frac{R}{2\sqrt{S}\sqrt{\chi}}\right]P(\chi)\right\} - \{\chi P''(\chi) + 2\chi P'(\chi)\} = 0. \tag{41}$$

For $S \to \infty$, we can neglect the last term of this equation and obtain the asymptotic form of the equation

$$\chi P''_{as}(\chi) + \left[a + 1 - \frac{a+b}{4S}\chi\right]P'_{as}(\chi) + \left[\frac{\epsilon + S^2 - \left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}+1\right)}{4S} - \frac{R}{2\sqrt{S}\sqrt{\chi}}\right]P_{as}(\chi) = 0. \tag{42}$$

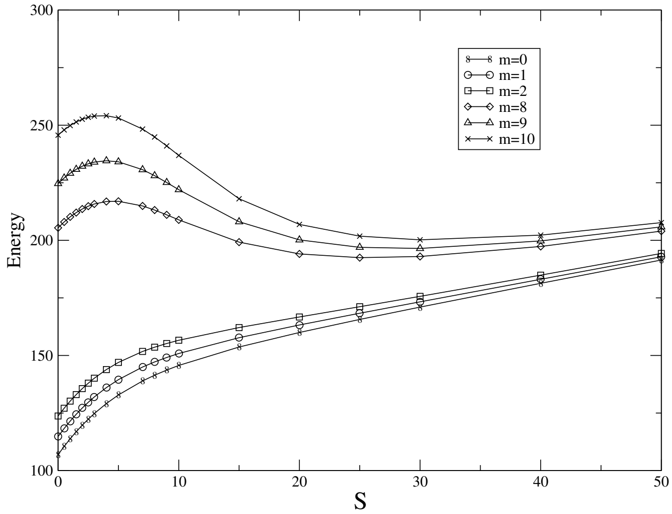

Figure 1. Energy for $n = 1$ versus S (proportional to B) for positive values of m.

With the last change of variable $\xi = \sqrt{\frac{\chi}{2}}$ and the fact that for $S \to \infty$, $a = |S-m| \to S$, $b = |S+m| \to S$, equation (42) reduces to a biconfluent Heun equation in its canonical form

$$\xi P_{as}''(\xi) + [1+\alpha-\beta\xi-2\xi^2]P_{as}'(\xi) + \left[(\gamma-\alpha-2)\xi - \frac{\delta+\beta(1+\alpha)}{2}\right]P_{as}(\xi) = 0 \qquad (43)$$

where α, β, γ and δ are expressed in terms of ϵ, S and $\frac{R}{l_0}$. As shown in [17], this equation describes the relative motion of two equal planar charged particles in a uniform perpendicular magnetic field and under the Coulomb repulsion. So the link between this problem and the planar problem is established and is locally acceptable when $R, S \to \infty$.

5. Results and comments

We present the results by giving the two first energy levels $n = 1, 2$ as a function of S, which is basically proportional to the constant magnetic field B. The value of the ratio R/l_0 is fixed for a given radius $R = 100l_0$. Figures 1 and 2 give the behaviour of the ground-state energy $\epsilon_{1,m}$ as a function of the applied magnetic field for positive and negative values of m.

We see clearly that these values are always bounded below as seen from classical considerations. For negative m values, the curves exhibit a sharp discontinuity in the slope of the tangent at some values of the magnetic field ($|m| = S$). There is also a more mild discontinuity in the derivative without change of sign for positive values of m.

In figure 3, the two situations are put together to display the phenomenon of level crossing with respect to the magnetic field. The behaviour of the level $\epsilon_{2,m}$ for positive values of m is given in figure 4. Finally crossovers between levels ϵ_1 and ϵ_2 for positive m are displayed in figure 5 . Note that at $B = 0$ we have the spectrum of the charged particle in the presence of pure repulsion centred at the north pole. For a strong magnetic field, $B \to \infty$, we can neglect the Coulomb repulsion, thus the levels tend to the Landau levels as expected.

Higher levels ($n > 2$) exhibit similar behaviour for $m > 0$ and $m < 0$ but calculations are more involved so they are not displayed here.

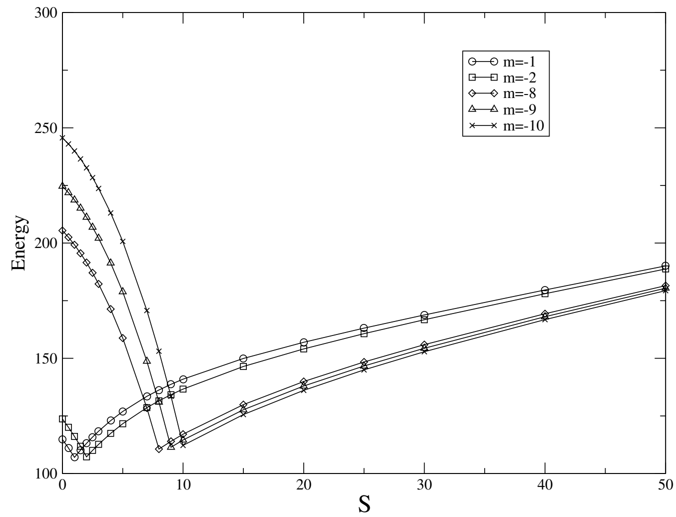

Figure 2. Energy for $n = 1$ versus S (proportional to B) for negative values of m.

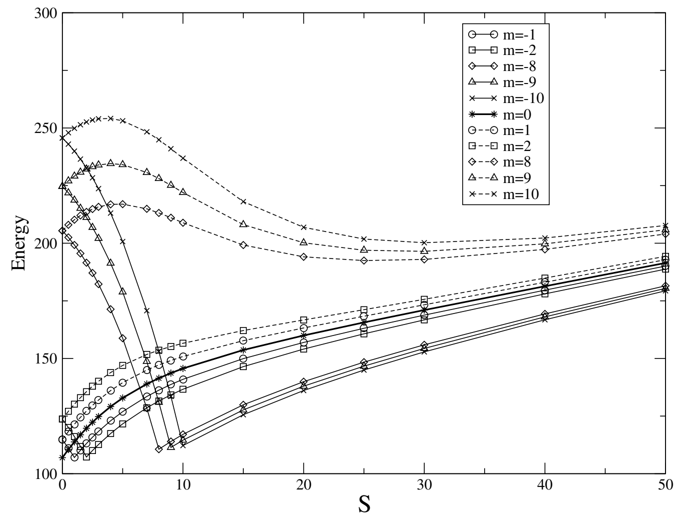

Figure 3. Energy for $n = 1$ versus S (proportional to B) for $m > 0$ and $m < 0$.

Let us examine now the limit of vanishing Coulomb repulsion ($l_0 \to \infty$). The energy levels of equation (38) become

$$\epsilon = \left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}+1\right) - \frac{R^4}{l_B^4} \tag{44}$$

and are no longer given by h_n. They are to be compared with the Landau levels on the sphere given by [1]

$$\epsilon = \left(n + \frac{a+b}{2}\right)\left(n + \frac{a+b}{2}+1\right) - S^2 \tag{45}$$

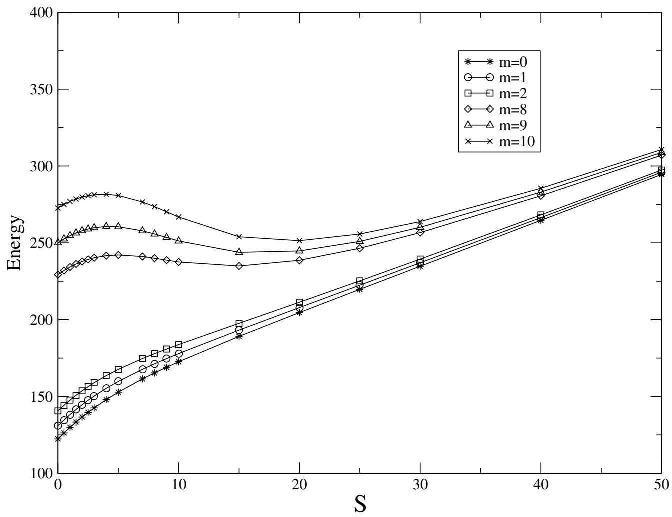

Figure 4. Energy for $n = 2$ versus S (proportional to B) for positive values of m.

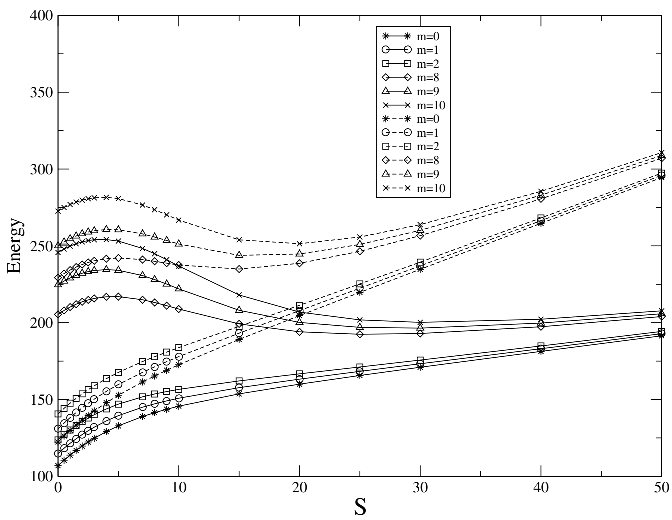

Figure 5. Crossover of the two first levels $\epsilon_{1,m}$ (full curve) and $\epsilon_{2,m}$ (dashed curves) versus S (proportional to B) for positive values of m.

where n is the Landau level index. Then, we recover the lowest Landau level ($n = 0$) on the sphere as expected previously. It is important to note that Haldane has considered only states with $a = S - m$, $b = S + m$ and $0 \leqslant m \leqslant S$.

To our knowledge, this is the first instance where a special class of Heun function, a solution of a linear differential equation with four singularities, occurs in quantum mechanics. This class of function can handle physical problems which carry three characteristic lengths. The spectrum energy presents different regimes, going from the pure Coulomb problem ($B = 0$) to the Landau problem ($B \to \infty$), passing through the well-known level mixing regime. This is the main difference with the pure magnetic problem [7–9] where, for weak fields, the energy levels are equally separated (Landau levels) instead of the Coulombic behaviour encountered here. Thus, the impurity plays an important role in the weak field regime but not in the high field regime. So, the impurity can be ignored for strong magnetic fields, i.e. for S beyond the sharp discontinuity at fixed m. The complete study of this wavefunction and its properties is left for a future work.

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